

RIGIDITY OF STABLE MINIMAL HYPERSURFACES IN ASYMPTOTICALLY FLAT SPACES

ALESSANDRO CARLOTTO

ABSTRACT. We prove that if an asymptotically Schwarzschildian 3-manifold (M, g) contains a properly embedded stable minimal surface, then it is isometric to the Euclidean space. This implies, for instance, that in presence of a positive ADM mass any sequence of solutions to the Plateau problem with diverging boundaries can never have uniform height bounds, even at a single point. An analogous result holds true up to ambient dimension seven provided polynomial volume growth on the hypersurface is assumed.

1. INTRODUCTION

Asymptotically flat manifolds naturally arise, in General Relativity, as models for *isolated gravitational systems* and can be regarded as one of the most basic classes of solutions to the Einstein equations. Their study has flourished over the last fifty years and the geometric and physical properties of these spaces have been widely investigated. In this article, our work is centered around the following fundamental question:

Problem (A). *Are there complete, stable minimal hypersurfaces in asymptotically flat manifolds?*

We specify here, once and for all, that the word *complete* is meant here to implicitly refer to *unbounded* minimal hypersurfaces.

Apart from their intrinsic relevance, stable minimal hypersurfaces naturally arise as limits of fundamental variational objects:

- (L1) sequences of minimizing currents solving the Plateau problem for diverging boundaries;
- (L2) sequences of large isoperimetric boundaries or, more generally, of large volume-preserving CMC hypersurfaces [EM12, EM13];

and thus their study plays a key role in the process of deeper understanding the large scale geometry of initial data sets.

From our perspective, the previous question had a twofold motivation: on the one end it could be regarded as a natural extension of the analysis of stable minimal hypersurfaces in the Euclidean space, on the other it implicitly arose in the proof of the *Positive Mass Theorem* by Schoen-Yau [SY79]. Even in the most basic of all cases, namely when (M, g) is \mathbb{R}^n with its flat metric and Σ^{n-1} is assumed to be an entire minimal graph, the study of (A) has played a crucial role in the development of Analysis along the whole course of the twentieth century:

Problem (B). *Are affine functions the only entire minimal graphs over \mathbb{R}^{n-1} in \mathbb{R}^n ?*

The author was partially supported by Stanford University and NSF grant DMS-1105323.

Indeed, minimal graphs are automatically stable (in fact: area-minimizing) by virtue of a well-known calibration argument [CM11] and thus (B) can be regarded as the most special subcase of (A). Such problem, which is typically named after S. N. Bernstein, was formulated around 1917 [Ber17] as an extension of the $n = 3$ case, which Bernstein himself had settled (see also [Fle62] for a different approach). In higher dimension, the answer is positive only up to ambient dimension 8 and is due to De Giorgi (for $n = 4$, [dG65]), Almgren (for $n = 5$, [Alm66]) and Simons (for $6 \leq n \leq 8$, [Sim68]) who also showed that the conjecture is false for $n \geq 9$ because of the existence of non-trivial area-minimizing cones in \mathbb{R}^{n-1} (see [BdGG69]).

When the ambient manifold is Euclidean, but Σ is only known to be stable (and not necessarily graphical) a similar classification result is only known when $n = 3$ and it was obtained independently by do Carmo and Peng [dCP79] and Fischer-Colbrie and Schoen [FS80]:

Theorem. [dCP79, FS80] *The only complete stable oriented minimal surface in \mathbb{R}^3 is the plane.*

However, the same statement is still not known to be true in \mathbb{R}^n for $n \geq 4$ unless the minimal hypersurface Σ^{n-1} under consideration is assumed to have *polynomial volume growth* meaning that for some (hence for any) point p

$$\mathcal{H}^{n-1}(\Sigma \cap B_r(p)) \leq \theta^* r^{n-1}, \quad \text{for all } r > 0.$$

Before stating our main theorems concerning question (A), we need to recall an essential physical assumption which will always be tacitly made in the sequel of this article. It is customary in General Relativity to assume that the energy density measured by any physical observer is *non-negative* at each point: this turns out to imply the requirement, in the time-symmetric case (which is the one we are considering here), that the scalar curvature be non-negative at all points of M .

Our first theorem states that there is a wide and physically relevant class of asymptotically flat manifolds for which the presence of a positive ADM mass is an obstruction to the existence of stable minimal surfaces. Once again, we refer the readers to the next section for the definitions of ADM mass: for the sake of this Introduction, they may consider the ADM mass \mathcal{M} a scalar quantity measuring the gravitational deformation of (M, g) from the trivial couple (\mathbb{R}^n, δ) .

Theorem 1. *Let (M, g) be an asymptotically Schwarzschildian 3-manifold of non-negative scalar curvature. If it contains a complete, properly embedded stable minimal surface Σ , then (M, g) is isometric to the Euclidean space \mathbb{R}^3 and Σ is an affine plane.*

An analogous result is obtained for ambient dimension $4 \leq n \leq 7$ under an a-priori bound on the volume growth of Σ (as we specified above) and provided the stability assumption is replaced by *strong stability*.

Theorem 2. *Let (M, g) be an asymptotically Schwarzschildian manifold of dimension $4 \leq n \leq 7$ and non-negative scalar curvature. If it contains a complete, properly embedded strongly stable minimal hypersurface Σ of polynomial volume growth, then (M, g) is isometric to the Euclidean space \mathbb{R}^n and Σ is an affine hyperplane.*

These two theorems also apply to the physically relevant case when the ambient manifold M has a compact boundary (an *horizon*, for instance) once it is assumed that $\Sigma \subset M \setminus \partial M$. If instead Σ is allowed to intersect the boundary ∂M (and thus to have a boundary $\partial\Sigma$), then we need to add extra requirements. For instance, when $n = 3$ we need Σ to be a *free boundary* minimal surface (with respect to $\partial\Sigma \subset \partial M$) and $\int_{\partial\Sigma} \kappa d\mathcal{H}^1 \geq -2\pi$.

We do not expect the assumption of *properness* to be inessential to the above theorems, unless Σ is assumed to be (locally) area-minimizing in which case properness can be easily proved via a standard local replacement argument. Moreover, we expect Σ to be proper whenever it arises as a limit of boundaries of isoperimetric domains: however, we will not investigate these aspects further in this work as we plan to analyse them carefully in a forthcoming paper with O. Chodosh and M. Eichmair.

Theorem 1 has several remarkable consequences and, among these, we would like to mention an application to the study of sequences of solutions to the Plateau problem for diverging boundaries belonging to a given hypersurface. As will be apparent from the statement, this can be interpreted as a result concerning the *failure of the convex hull property* in asymptotically flat spaces. We will first need some terminology. Given an asymptotically flat manifold (M, g) with one end and, correspondingly, a system of asymptotically flat coordinates $\{x\}$ we call *hyperplane* a subset of the form $\Pi = \{x \in \mathbb{R}^n \setminus B \mid \sum_{i=1}^n a_i x^i = 0\}$ for some real numbers a_1, a_2, \dots, a_n . Possibly by changing $\{x\}$ (we do not rename) one can always reduce to the case when $a_1 = \dots = a_{n-1} = 0$ and $a_n = 1$. In this setting, we define *height* of a point in $M \setminus Z \simeq \mathbb{R}^n \setminus B$ the value of its x^n -coordinate. Moreover, we denote by x' the ordered $(n-1)$ -tuple corresponding to the first $(n-1)$ coordinates of a point in $M \setminus Z$.

Corollary 3. *Let (M, g) be an asymptotically Schwarzschildian 3-manifold of non-negative scalar curvature and positive ADM mass, let Π an hyperplane and let $(\Omega_i)_{i \in \mathbb{N}}$ any monotonically increasing sequence of regular, relatively compact domains such that $\cup_i \Omega_i = \Pi$. For any index i , define Γ_i to be a solution of the Plateau problem with boundary $\partial\Omega_i$. Then for each $x' \in \Pi$ the sequence $(\Gamma_i)_{i \in \mathbb{N}}$ cannot have uniformly bounded height at x' , namely*

$$\liminf_{i \rightarrow \infty} \max_{(x', x^3) \in \Gamma_i} x^3 = +\infty.$$

This corollary follows at once from Theorem 1 by means of a standard compactness argument (see, for instance, [Whi87b]).

As anticipated above, our proof of Theorem 1 is strongly inspired by the proof given by Schoen-Yau of the *Positive Mass Theorem* in [SY79] where (arguing by contradiction) negativity of the ADM mass is exploited for constructing a (strongly) stable complete minimal surface of planar type, thereby violating the stability inequality by a preliminary reduction, via a density argument, to a Riemannian metric of strictly positive scalar curvature, at least outside a compact set. In our case, we need to deal with two substantial differences:

- (1) the hypersurface Σ is not constructed but is just assumed to exist and thus its structure and its behaviour at infinity are not known a priori;
- (2) the metric g is only required to have non-negative scalar curvature, thereby admitting the (relevant) case when it is in fact scalar flat, as prescribed by the Einstein constraints in the *vacuum* case.

One crucial part of our study (and a preliminary step in the proof of Theorem 1 and Theorem 2) is indeed to characterize the structure at infinity of a complete minimal hypersurface

having finite Morse index. Essentially, we extend to asymptotically flat manifolds the Euclidean structure theorem by Schoen [Sch83] which states, roughly speaking, that any such minimal hypersurface has to be *regular at infinity* in the sense that it can be decomposed (outside a compact set) as a finite union of graphs with at most logarithmic growth when $n = 3$ or polynomial decay (like $|x'|^{3-n}$) when $n \geq 4$. Schoen proved this theorem making substantial use of the Weierstrass representation for minimal hypersurfaces, a tool which is strongly peculiar of the Euclidean setting as its applicability relies on the fact that the coordinate function have harmonic restrictions to minimal submanifolds. In our case, a key step in the proof of Theorem 1 is proving (via a preliminary study of the limit laminations which may arise by blow-down of Σ) that any such Σ has finite total curvature and hence its Gauss map extends continuously at infinity. Instead, such an argument is not at disposal in the higher-dimensional case and hence a deep result of L. Simon [Sim83b, Sim85] concerning the analysis of isolated singularities of minimal subvarieties has to be used.

In fact, Theorem 1 and Theorem 2 follow from a more general rigidity result concerning marginally outer-trapped hypersurfaces (or MOTS), a class of objects of fundamental importance in General Relativity and which coincide with minimal hypersurfaces for time-symmetric data. This extension is not trivial since MOTS are not known to have a variational nature (namely they are not known to arise as critical points of a functional) and thus a completely different, and substantially more involved approach must be followed [Car14].

The results contained in this article have been first announced in June 2013 and made available, in a preliminary form, in October 2013: in that version we gave a rather different proof of Theorem 1 which seemed to require a quadratic area growth assumption, in analogy with the higher dimensional case. While completing the preparation of the present article, we were communicated that O. Chodosh and M. Eichmair were able to remove, independently of us, such assumption. Their proof introduced an interesting shortcut with respect to our approach by considering the different decay rate of the terms in the Gauss equation for Σ , which allows to avoid computing the precise asymptotic expansion of the scalar curvature of Σ . However, this is based on the fact that the Schwarzschild metric is scalar flat, while in more general situations (like those we address in [Car14]) our computations seem quite necessary.

We would like to conclude this Introduction by mentioning the forthcoming paper [CS14] by the author and R. Schoen, where we show that the rigidity Theorem 1 (as well as Theorem 2) is essentially sharp by constructing asymptotically flat solutions of the Einstein constraint equations in \mathbb{R}^n (for $n \geq 3$) that have positive ADM mass and are *exactly flat* outside of a solid cone (for any positive value of the corresponding opening angle) so that they contain plenty of complete, area-minimizing hypersurfaces. A posteriori, this strongly justifies our requirement that the metric g is asymptotically Schwarzschildian. Moreover, such *flexibility* result turns out to clarify some deep phenomena concerning the role played by the scalar curvature in the geometry of asymptotically flat spaces and, from a different perspective, allows the construction of new classes of N -body solutions of the Einstein constraint equations.

Acknowledgments. The author wishes to express his deepest gratitude to his PhD advisor Richard Schoen for his outstanding guidance and for his constant encouragement: his influence on this work is more than manifest. He would also like to thank Simon Brendle, Alessio Figalli, Andrea Malchiodi, Rafe Mazzeo, Andras Vasy, Brian White and especially

Otis Chodosh and Michael Eichmair for several useful discussions and for their interest in this work.

2. DEFINITIONS AND NOTATIONS

We need to start by recalling the definition of weighted Sobolev and Hölder spaces in the Euclidean setting (\mathbb{R}^n, δ) . Here and in the sequel we always assume $n \geq 3$ and we let $r \in \mathcal{C}^\infty(\mathbb{R}^n)$ any positive function which equals the usual Euclidean distance $|\cdot|$ outside of the unit ball.

Definition 2.1. Given an index $1 \leq p \leq \infty$ and a weight $\delta \in \mathbb{R}$ we define the weighted Lebesgue spaces $\mathcal{L}_\delta^p(\mathbb{R}^n)$ as the sets of measurable functions on \mathbb{R}^n such that the corresponding norms $\|\cdot\|_{\mathcal{L}_\delta^p}$ are finite, with

$$\|u\|_{\mathcal{L}_\delta^p} = \begin{cases} \left(\int_{\mathbb{R}^n} |u|^p r^{-\delta p - n} d\mathcal{L}^n \right)^{1/p}, & \text{if } p < \infty \\ \|r^{-\delta} u\|_{\mathcal{L}^\infty}, & \text{if } p = \infty. \end{cases}$$

Correspondingly, for an integer $k \in \mathbb{N}$ we define the weighted Sobolev spaces $\mathcal{W}_\delta^{k,p}(\mathbb{R}^n)$ as the sets of measurable functions for which the norms

$$\|u\|_{\mathcal{W}_\delta^{k,p}} = \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{\mathcal{L}_\delta^{p-|\alpha|}}$$

are finite.

Definition 2.2. Given an integer $k \in \mathbb{N}$, and real numbers $\delta \in \mathbb{R}$ and $\alpha \in (0, 1)$ we define the Hölder space $\mathcal{C}_\delta^{k,\alpha}(\mathbb{R}^n)$ as the set of continuous functions on \mathbb{R}^n for which the norm $\|\cdot\|_{\mathcal{C}_\delta^{k,\alpha}}$

$$\|u\|_{\mathcal{C}_\delta^{k,\alpha}} = \sum_{0 \leq |\gamma| \leq k} \sup_{x \in \mathbb{R}^n} r(x)^{-\delta-|\gamma|} |\partial^\gamma u(x)| + \sum_{|\gamma|=k} \sup_{|x-y| \leq r(x)} \frac{|r(x)^{-\delta-k} \partial^\gamma u(x) - r(y)^{-\delta-k} \partial^\gamma u(y)|}{|x-y|^\alpha}$$

is finite.

If M is a \mathcal{C}^k (resp. $\mathcal{C}^{k,\alpha}$) manifold such that there is a compact set Z for which $M \setminus Z$ consists of a finite, disjoint union $\bigsqcup_{l=1}^N E_l$ and for each index l there is a diffeomorphism (of the appropriate level of regularity) $\Phi_l : E_l \rightarrow \mathbb{R}^n \setminus B_l$ (for some Euclidean ball B_l) then one can easily define the weighted spaces $\mathcal{W}_\delta^{k,p}$ (resp. $\mathcal{C}_\delta^{k,\alpha}$). That is done, routinely, by choosing a finite atlas of M consisting of the charts at infinity together with finitely many pre-compact charts and adding the $\mathcal{W}_\delta^{k,p}(\mathbb{R}^n \setminus B_l)$ (resp. $\mathcal{C}_\delta^{k,\alpha}(\mathbb{R}^n \setminus B_l)$) norm on the former to the $\mathcal{W}^{k,p}$ (resp. $\mathcal{C}^{k,\alpha}$) norm on the latter ones. The resulting spaces $\mathcal{W}_\delta^{k,p}$ (resp. $\mathcal{C}_\delta^{k,\alpha}$) as well as their topology are not canonically defined, yet only depend on the choice of the diffeomorphisms Φ_l for $l = 1, \dots, N$. Such definition is easily extended to tensors of any type by following the very same pattern.

Definition 2.3. Given an integer $n \geq 3$, and real numbers $p > \frac{n-2}{2}$, $\alpha \in (0, 1)$ a complete manifold (M, g) is called asymptotically flat of type (p, α) if:

- (1) there exists a compact set $Z \subset M$ (the *interior* of the manifold) such that $M \setminus Z$ consists of a disjoint union of *finitely many* ends, namely $M \setminus Z = \bigsqcup_{l=1}^N E_l$ and for each index l there exists a smooth diffeomorphism $\Phi_l : E_l \rightarrow \mathbb{R}^n \setminus B_l$ for some open

ball $B_l \subset \mathbb{R}^n$ containing the origin so that the pull-back metric $(\Phi_l^{-1})^* g$ satisfies the following condition:

$$((\Phi_l^{-1})^* g)_{ij} - \delta_{ij} \in \mathcal{C}_{-p}^{2,\alpha}(\mathbb{R}^n \setminus B_l)$$

(2) the *mass density* μ is integrable, namely

$$\mu \in \mathcal{L}^1(M).$$

As first suggested in [SY79], (but see also [EHLS11]) for a number of purposes it is convenient to work, whenever possible, with asymptotically flat data that have a particularly simple description at infinity.

Definition 2.4. Let $n \geq 3$, and let (M, g) be an *asymptotically flat* manifold. We say that (M, g) is asymptotically Schwarzschildian if there exists diffeomorphisms (as in the Definition 2.3) as well as (for each end) a function $h \in \mathcal{C}_{2-n}^{2,\alpha}$ such that for $i, j = 1, 2, \dots, n$

$$\begin{aligned} h(x) &= 1 + a|x|^{2-n} \\ g_{ij} &= h^{\frac{4}{n-2}} \delta_{ij} + O^{2,\alpha}(|x|^{1-n}). \end{aligned}$$

We then recall the notion of ADM mass, which was introduced in [ADM59] in the context of Hamiltonian formulation of General Relativity.

Definition 2.5. Given an asymptotically flat manifold (M, g) (so that μ is integrable) one can define the ADM *mass* \mathcal{M} at each end to be

$$\mathcal{M} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{|x|=r} \sum_{i,j=1}^n (g_{ij,i} - g_{ii,j}) \nu_0^j d\mathcal{H}^{n-1}$$

where we have set $\nu_0^j = \frac{x^j}{|x|}$ and ω_{n-1} is the volume of the standard unit sphere in \mathbb{R}^n . In case of multiple ends, these quantities can be defined by additive extension.

In 1979 Schoen and Yau proved the most fundamental property of this quantity, namely its positivity.

Theorem 2.6. [SY79, Wit81] *Let (M, g) be an asymptotically flat manifold of dimension $3 \leq n < 8$ satisfying the dominant energy condition. Then $\mathcal{M} \geq 0$ and equality holds if and only if (M, g) is isometric to the Euclidean space (\mathbb{R}^n, δ) .*

In fact, a closer look at the proof of the equality case [Sch84] shows that in case of multiple ends, it is enough to have *one* end of null ADM mass to force the whole space to be globally isometric to (\mathbb{R}^n, δ) .

Our rigidity results need to make use of the physical meaning of the constant a given in Definition 2.4). To that aim, we recall the following basic computation (the reader might check it, for example, in [EHLS11]).

Lemma 2.7. *Let (M, g) be an asymptotically flat manifold (with one end) having harmonic asymptotics and let a be given by Definition 2.4. Then*

$$\mathcal{M} = \frac{(n-2)}{2}a.$$

As a result, if we prove that an asymptotically Schwarzschild manifold (of non-negative scalar curvature) has $a = 0$ then it follows by the *Positive Mass Theorem* 2.6 that the expansion of the metric has to be trivial to *all* orders (namely $g = \delta$).

The very same computation shows, more generally, the following: if (M_1, g_1) is an asymptotically flat manifold (according to Definition 2.3) and $h = 1 + a|x|^{2-n} + O^{2,\alpha}(|x|^{1-n})$ is $\mathcal{C}_{loc}^{2,\alpha}$ then the ADM mass of (M_2, g_2) where $g_2 = h^{\frac{4}{n-2}}g_1$ and n is the dimension of M_1 is given by

$$(2.1) \quad \mathcal{M}_2 = \mathcal{M}_1 - \frac{1}{2\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{|x|=r} h \nabla_{\nu_1} h d\mathcal{H}^{n-1} = \mathcal{M}_1 + \frac{(n-2)}{2}a.$$

Notations. We denote by R (resp. $Ric(\cdot, \cdot)$) the scalar (resp. Ricci) curvature of (M, g) , by R_Σ the scalar curvature of $\Sigma \hookrightarrow (M, g)$ and by ν (a choice of) its unit normal. For Euclidean balls centered at the origin we omit explicit indication of the center and hence we write B_r instead of $B_r(p)$. We let C be a real constant which is allowed to vary from line to line, and we specify its functional dependence only when this is relevant or when ambiguity is likely to arise.

3. PROOF OF THE RIGIDITY STATEMENTS

The scope of this section is to give a detailed proof of Theorem 1 and Theorem 2. Our arguments turn out to be substantially different in the case $n = 3$ (treated in Section 3.1) and $4 \leq n \leq 7$ (treated in Section 3.2), the former being more self-contained and elementary in nature.

We introduce a viewpoint which will turn out to be very convenient both here and, in [Car14], when dealing with marginally outer-trapped hypersurfaces. Given $\Sigma \hookrightarrow M$ a minimal hypersurface in an asymptotically flat manifold (Definition 2.3) we let Σ^i be a connected component of $\Sigma \setminus Z$, where Z is the core of M . Thus, there exists an end E_j of M such that $\Sigma^i \hookrightarrow E_j$ and so (by means of the diffeomorphism $\Phi_j : E_j \rightarrow \mathbb{R}^n \setminus B_j$) we can consider a copy of Σ^i , which we will call Σ_0^i , as a submanifold with boundary in $\mathbb{R}^n \setminus B_j$ with the geometry induced by the ambient Euclidean metric. As a result, each Σ_0^i is not minimal, but is a stationary point of a functional $\mathbf{F} = \mathbf{F}_0 + \mathbf{E}$ with \mathbf{F}_0 the $(n-1)$ -dimensional Hausdorff measure and \mathbf{E} an error term which decays at infinity with a rate that depends on the asymptotics of $g_{ij} - \delta_{ij}$ at infinity. Moreover, if Σ is stable then each Σ_0^i will be a stable stationary point for \mathbf{F} . Finally, it is convenient to assume that for each index j the ball B_j is centered at the origin (which of course we can always arrange, by means of a translation). We denote the ADM mass of the end E_j by \mathcal{M}_j and by \mathcal{M} the total mass of M .

3.1. The proof for $n = 3$. In the setup described above, let us assume to fix an index i and so we will denote, for simplicity of notation, Σ_0^i by $\Sigma_0 \hookrightarrow \mathbb{R}^3 \setminus B$. Let us denote by A_0 the second fundamental form of Σ_0 , by H_0 (resp. K_0) its mean (resp. Gaussian) curvature and by ν_0 its Euclidean Gauss map. We divide the proof in a few steps, with some results of independent interest.

Proof. Step 1: annular decomposition.

By means of a variation on standard curvature estimates in [Sch83b], we know that there exists a constant $C > 0$ such that (for some, hence for any, fixed point $p \in M$)

$$\sup_{p \in \Sigma \setminus Z} d_g(p, p_0) |A(p)| \leq C, \quad \sup_{p \in \Sigma} |A(p)| \leq C$$

and thus, by virtue of the simple comparison result

$$|A_0(x) - A(x)| \leq \frac{C}{|x|^2} (|x||A(x)| + 1)$$

(where $\{x\}$ is a set of asymptotically flat coordinates in E_j) we have that

$$|A_0(x)| \leq \frac{C}{|x|}.$$

Similarly, since Σ is assumed to be minimal one obtains that $|H_0(x)| \leq C|x|^{-2}$.

These two facts imply that for any sequence $\lambda_m \searrow 0$ the rescaled surfaces $\lambda_m \Sigma_0$ converge (up to a subsequence, which we do not rename) to a *stable minimal lamination* \mathcal{L} in $\mathbb{R}^3 \setminus \{0\}$. The convergence happens, locally, in the sense of smooth graphs.

Now, let L be a leaf, namely a (maximal) connected component of \mathcal{L} . If $0 \notin \overline{L}$ in \mathbb{R}^3 , then $L = \overline{L}$ is a connected, stable minimal surface in the Euclidean space, hence a plane by [FS80, dCP79]. However, the same conclusion is also true in the case when $0 \in \overline{L}$ thanks to a removable singularity theorem obtained by Gulliver and Lawson [GL86] (see also Meeks-Perez-Ros [MPR13] and Colding-Minicozzi [CM05]). As a result, every leaf of \mathcal{L} is a flat plane in \mathbb{R}^3 and hence, since trivially any two leaves cannot intersect (due to the very definition of lamination) we conclude that, modulo an ambient isometry, $\mathcal{L} = \mathbb{R}^2 \times Y$ for some $Y \subset \mathbb{R}$ closed.

Because a plane in \mathbb{R}^3 is totally geodesic, it follows at once that we can upgrade our curvature estimate to $|A_0(x)| \leq o(1)|x|^{-1}$ as x goes to infinity. This implies that for r_0 large enough the surface $\Sigma_0 \setminus B_{r_0}$ (which, we recall, is assumed to be proper) satisfies $|A_0(x)| \leq C|x|^{-1}$ for some $C \in (0, 1)$ and we claim this forces $\Sigma_0 \setminus B_{r_0}$ to be annular.

Indeed, let $f : \Sigma_0 \rightarrow \mathbb{R}$ be the restriction of the Euclidean distance function $|\cdot|$ to the complete surface Σ . The critical points of f occur at those points x_0 where the unit normal ν_0 is parallel to the position vector x . A trivial computation shows that the Hessian at such a point is given by $\nabla^2 f(x_0)[v, v] = 2(|v|^2 - A_0(x_0)[v, v]\delta(x_0, \nu_0))$ for any vector $v \in T_{x_0}\Sigma_0 \simeq \mathbb{R}^2$. We claim that indeed, under our assumptions $\nabla^2 f(x_0)[v, v] > 0$ for all $v \in T_{x_0}\Sigma_0$ and, to check that, it is enough (by homogeneity) to reduce to the case when v has unit (Euclidean) length. This is trivial, since $\nabla^2 f(x_0)[v, v] \geq 2(1 - |A_0(x_0)||x_0|) \geq 2(1 - C) > 0$. That shows that all *interior* critical points of the function f are strict local minima, which implies (since Σ_0 is connected) that f does not have any interior critical points at all and so Σ_0 is annular, as we had to prove. By working one annular component at a time we can then assume, without renaming, that Σ_0 is a connected annulus.

Step 2: finiteness of the total curvature.

Let us observe that, since $\lambda_m \searrow 0$ it is trivially true that $\mathbb{R}^2 \times \{0\}$ is a leaf of the foliation \mathcal{L} : we claim that it has to be the only one. Indeed, the upgraded estimate $|A_0(x)| \leq o(1)|x|^{-1}$ implies, by means of a standard graphicality argument (see e. g. Lemma 2.4 in [CM11]) that for any $\sigma > 0$ there exists an index m_0 large enough that when $m \geq m_0$ the surface $\lambda_m \Sigma_0$ has a graphical component in the region $B_{3\sigma/2} \setminus B_{\sigma/2}$ with a defining function f with converges to 0 in \mathcal{C}^2 . If there were another leaf in \mathcal{L} , that should have the form $\mathbb{R}^2 \times \{y_0\}$ for some $y_0 \in \mathbb{R}^\times$ and hence $\lambda_m \Sigma_0$ should have another graphical component in the annulus $B_{|y_0|/4}((0, 0, y_0)) \setminus B_{|y_0|/8}((0, 0, y_0))$, also with a defining function f_0 whose Lipschitz constant converges to zero as $m \rightarrow \infty$. As a result of these two statements (the former taken for

$\sigma = |y_0|$), the (Euclidean) sphere centered at the origin and having radius $7|y_0|/6$ should meet $\lambda_m \Sigma$ along two circles, at least for large m . On the other hand, we know by the previous argument that $\lambda_m \Sigma_0$ is an annulus, and thus we get a contradiction. This leads to the conclusion that in fact $\mathcal{L} = \mathbb{R}^2 \times \{0\}$, a single leaf. We claim that this also forces the total curvature of Σ_0 to be finite.

If not, we could find an increasing sequence of radii $r_m \nearrow \infty$ (with $r_m > r_0$ for each m) such that the total curvature of the annulus $\Sigma \cap (B_{r_m} \setminus B_{r_0})$ is more than m . Hence, by the Gauss-Bonnet theorem applied to this annuli, the contributions of the geodesic curvature of the outer boundaries should also become arbitrarily large as $m \rightarrow \infty$. We could then consider the rescaled sequence gotten by taking $\lambda_m = r_m^{-1}$ and we should have, on the one hand

$$\int_{\lambda_m \Sigma \cap \partial B_1} |\kappa| d\mathcal{H}^1 > \frac{m}{2}$$

(at least for m large enough), while on the other we know that $\lambda_m \Sigma \cap (B_{3/2} \setminus B_{1/2})$ is a graph with small Lipschitz constant, hence

$$\int_{\lambda_m \Sigma \cap \partial B_1} |\kappa| d\mathcal{H}^1 \leq 3\pi$$

thus reaching a contradiction.

Step 3: expansion at infinity.

By virtue of the previous step, the surface Σ_0 has finite total curvature, hence by Theorem 1 and Theorem 3 in [Whi87] we conclude that Σ_0 is conformally diffeomorphic to a punctured disk and the corresponding (Euclidean) Gauss map extends to the puncture. This means that, by possibly taking a larger value of r_0 and applying an ambient isometry, the surface Σ_0 coincides with the graph of a smooth function $u : \Pi \rightarrow \mathbb{R}$ (for some plane Π) such that

$$(3.1) \quad \lim_{r \rightarrow \infty} |\nabla u| = 0, \quad \int_{\Pi} |\nabla \nabla u|^2 d\mathcal{L}^2 < \infty.$$

Before proceeding further, we need to add two important comments about the applicability of the aforementioned Theorem 3 in [Whi87]. First: one needs to check that $\int_{\Sigma_0} |A_0|^2 d\mathcal{H}^2$, rather than $\int_{\Sigma_0} |K_0| d\mathcal{H}^2$ which we already know. But this comes from the fact $|A_0|^2 = H_0^2 - 2K_0$ together with the decay $|H_0(x)| \lesssim |x|^{-2}$ provided we observe that Σ_0 has quadratic area growth, which in turn follows (via the co-area formula) by showing that

$$\mathcal{H}^1(\Sigma_0 \cap \partial B_r) \leq Cr, \quad \text{for all } r > 0.$$

This is established by a blow-down argument which is essentially identical to the one we gave above. Second: we need to check that the Gauss curvature of Σ_0 is non-positive, at least outside a (possibly larger) compact set of \mathbb{R}^3 . To this aim, it is more convenient to consider $\Sigma \hookrightarrow M$ instead. Recall that

$$\text{Ric}(\nu, \nu) + \frac{1}{2}|A|^2 = \frac{1}{2}R - K$$

and for our class of data (see Definition 2.4) we have

$$R \leq C|x|^{-4}$$

(because the Schwarzschild metric is itself scalar flat) and if $\mathcal{M}_j > 0$

$$Ric(\nu, \nu) \geq C|x|^{-3}$$

due to the general formula for the Ricci tensor

$$Ric_{kl} = \frac{\mathcal{M}_j}{|x|^3} \left(1 + \frac{\mathcal{M}_j}{2|x|}\right)^{-2} \left(\delta_{kl} - 3 \frac{x^p x^q}{|x|^2} \delta_{kp} \delta_{lq}\right) + O(|x|^{-4})$$

and (again) the fact that $g(\nu, \partial_r) < 1/3$ for r large enough, which follows from the above arguments. So these two remarks imply the results in [Whi87] are applicable and so we have a global graphical description of Σ_0 . Moreover, we can then exploit the rough information (3.1) to improve our decay estimate to $u(x') \lesssim |x'|^{1-\alpha}$ (for $\alpha > 0$) and hence use this in the (perturbed) minimal surface equation to get, by a standard elliptic bootstrap argument, that in fact

$$u(x') = a \log |x'| + b + O(|x'|^{-1})$$

(the details are discussed in Appendix A). This shows that Σ_0 and hence Σ is regular at infinity, in the sense of Schoen [Sch83] (a result which he proved in \mathbb{R}^3 using the Weierstrass representation).

Step 4: infinitesimal rigidity.

Now, we are going to exploit all of this information about the behaviour of Σ at infinity. Indeed, the stability inequality (with the rearrangement trick by Schoen-Yau) takes the form

$$\frac{1}{2} \int_{\Sigma} (R + |A|^2) \phi^2 d\mathcal{H}^2 \leq \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 d\mathcal{H}^2 + \int_{\Sigma} K \phi^2 d\mathcal{H}^2$$

so by means of the logarithmic cut-off trick

$$\frac{1}{2} \int_{\Sigma} (R + |A|^2) d\mathcal{H}^2 \leq \int_{\Sigma} K d\mathcal{H}^2$$

and thanks to the Gauss-Bonnet theorem for open manifolds (see Shiohama [Shi85] or [Whi87]) we know that

$$\int_{\Sigma} K d\mathcal{H}^2 = 2\pi(\chi(\Sigma) - N')$$

(where $N' \geq 1$ is the number of ends of Σ) hence

$$0 \leq \frac{1}{2} \int_{\Sigma} (R + |A|^2) \phi^2 d\mathcal{H}^2 \leq 2\pi(\chi(\Sigma) - N')$$

which forces $\chi(\Sigma) = 1$, $N' = 1$ and Σ to be totally geodesic and with vanishing restriction of the ambient scalar curvature. An argument by Fischer-Colbrie and Schoen [FS80] gives that Σ is intrinsically flat.

Step 5: conclusion via positive mass theorem.

To conclude, we recall that if the ADM mass of E_j in (M, g) is nonzero, then the Gauss curvature of Σ is negative (at least far away from the core) so this contradicts the conclusion of the previous step. Thus the mass in question has to be equal to zero, and hence by Theorem 2.6 we conclude that (M, g) is the Euclidean space \mathbb{R}^3 which completes the proof. \square

Remark 3.1. One could shorten the first part of the proof by observing that, based on Theorem 3 in [FS80] the surface Σ has to be conformally diffeomorphic to a plane and hence finiteness of the total curvature follows from the stability inequality together with the conformal invariance of the Dirichlet integral. However, that approach would make use of the *global* stability assumption, while we only made use of *outer stability* of Σ (which is equivalent to finiteness of its Morse index) and thus proved a more general statement about the structure at infinity of finite index minimal surfaces in asymptotically flat spaces.

3.2. The proof for $4 \leq n \leq 7$. We now move to the proof of Theorem 2, namely the higher dimensional counterpart of the previous one. It is convenient to recall here the notion of strong stability.

Definition 3.2. Given $\alpha \in \mathbb{R}$ we set

$$(3.2) \quad \mathcal{V}_\alpha(\Sigma) = \left\{ \phi + \alpha \mid \phi \in \mathcal{W}_{\frac{3-n}{2}}^{1,2}(\Sigma) \right\},$$

and we also define

$$(3.3) \quad \mathcal{V}(\Sigma) = \bigcup_{\alpha \in \mathbb{R}} \mathcal{V}_\alpha(\Sigma).$$

We say that a minimal hypersurface $\Sigma \hookrightarrow (M, g)$ is strongly stable if the stability inequality

$$\int_{\Sigma} (Ric(\nu, \nu) + |A|^2) \phi^2 d\mathcal{H}^{n-1} \leq \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 d\mathcal{H}^{n-1}$$

is true for any test function $\phi \in \mathcal{V}$.

When Σ is known a priori to approach, at suitably good rate, an hyperplane along one of its ends the previous notion has a natural geometric interpretation: Σ is strongly stable if it is stable with respect to all deformations that are essentially *vertical translations* near infinity.

The proof of this theorem follows a conceptual scheme which is quite similiar to the one followed to prove Theorem 1, with the substantial difference that we make use of a deep result of L. Simon [Sim83b, Sim85] concerning the uniqueness of tangent cones in the context of his study of isolated singularities of geometric variational problems.

Proof. We work here with the very same notations defined above and so let Σ_0 be (with slight abuse of notation) one connected component of $\Sigma \setminus Z$, and considered as a properly embedded submanifold of $\mathbb{R}^n \setminus B$ for some Euclidean ball B centered at the origin. Of course $\partial \Sigma_0 \subset \partial B$.

Step 1: annular decomposition.

First of all, let us observe that (if $\{x\}$ is a set of Euclidean coordinates in \mathbb{R}^n) then the integral

$$\int_{\Sigma_0} \frac{|x^\perp|^2}{|x|^{n+1}}$$

is finite (here $x^\perp = \delta(x, \nu_0)$, the projection of the position vector onto the normal space of Σ_0 at the point in question). Such claim easily follows from the general monotonicity formula

by Allard (see also Section 17 in [Sim83]), true for any integral varifold V of bounded mean curvature H_0

$$\frac{\mu_V(B_\rho(\xi))}{\rho^k} - \frac{\mu_V(B_\sigma(\xi))}{\sigma^k} = \int_{B_\rho(\xi)} \frac{H_0}{k} \cdot (x - \xi) \left(\frac{1}{m(r)^k} - \frac{1}{\rho^k} \right) d\mu_V + \int_{B_\rho(\xi) \setminus B_\sigma(\xi)} \frac{|\partial^\perp r|^2}{r^k} d\mu_V$$

(where $m(r) = \max\{r, \sigma\}$): indeed, in our case the left-hand side is bounded because of the polynomial volume growth assumption, and of course $\int_{\Sigma_0} \frac{|H_0|}{|x|^{n-2}} d\mathcal{H}^{n-1}$ is finite because $|H_0(x)| \lesssim |x|^{-2}$ since $H = 0$ identically, by minimality of Σ .

We then claim that the previous can be turned into a pointwise decay estimate, in the sense that $|x||x^\perp| = o(1)$ as $|x|$ goes to infinity. To that aim, we argue as follows. Suppose, by contradiction, that such statement were false: then we could find $0 < \varepsilon < 1$ and a sequence of points $(x_i)_{i \in \mathbb{N}}$ belonging to Σ_0 such that the following two conditions hold:

$$(3.4) \quad \begin{cases} |x_i| \nearrow \infty, \text{ as } i \rightarrow \infty \\ \frac{|x_i^\perp|}{|x_i|} \geq \varepsilon, \text{ for all } i \in \mathbb{N}. \end{cases}$$

Now, observe that since Σ is stable we know by making use of Theorem 3 in Section 6 of [SS81] that there exists a constant $C > 0$ such that

$$\sup_{p \in \Sigma \setminus Z} d_g(p, p_0) |A(p)| \leq C, \quad \sup_{p \in \Sigma} |A(p)| \leq C$$

and, once again, by the comparison result $|A_0(x) - A(x)| \leq C \frac{1}{|x|^2} (|x||A(x)| + 1)$ we conclude $|A_0(x)| \leq \frac{C}{|x|}$.

This immediately implies that $|\nabla_{\Sigma_0}(x \cdot \nu_0)| \leq C|x||A_0(x)| \leq C$ for some constant $C \geq 1$. This gradient bound implies that if $r = |x|$ is large enough and $|x \cdot \nu_0(x)| \geq \varepsilon r$ then one should also have $|y \cdot \nu_0(y)| \geq \varepsilon r/2$ at all points $y \in \Sigma_0$ belonging to the following set:

$$\mathcal{K}_x = \left\{ y \in \Sigma_0 \mid \text{there exists a path } \gamma_{x,y} : [0, 1] \rightarrow \Sigma_0 \text{ with } \gamma(0) = x, \gamma(1) = y, \text{ length}(\gamma_{x,y}) \leq \frac{\varepsilon r}{2C} \right\}$$

where $\text{length}(\gamma)$ denotes the length of the path γ and C is the constant defined above. We claim that in fact the set \mathcal{K}_x should contain the (extrinsic) ball of center x and radius $\varepsilon r/(4C)$. This follows by a standard graphicality argument (see, for instance Lemma 2.4 in [CM11]) again thanks to the pointwise decay assumption on the second fundamental form of Σ_0 . If we apply this argument to each of the points x_i keeping in mind their definition (see (3.4)) we get that

$$\begin{aligned} \int_{\Sigma_0 \cap \{|x| > r_i/2\}} \frac{|y \cdot \nu_0(y)|^2}{|y|^{n+1}} d\mathcal{H}^{n-1}(y) &\geq \int_{\mathcal{K}_{x_i}} \frac{|y \cdot \nu_0(y)|^2}{|y|^{n+1}} d\mathcal{H}^{n-1}(y) \\ &\geq \int_{B_{\left(\frac{\varepsilon r_i}{4C}\right)}(x_i)} \frac{|y \cdot \nu_0(y)|^2}{|y|^{n+1}} d\mathcal{H}^{n-1}(y) \geq C\varepsilon^{n+1} \end{aligned}$$

(for a suitable constant C) but on the other hand we already know, that it must be

$$\lim_{i \rightarrow \infty} \int_{\Sigma_0 \cap \{|x| > r_i/2\}} \frac{|y \cdot \nu_0(y)|^2}{|y|^{n+1}} d\mathcal{H}^{n-1}(y) = 0$$

and these two facts together give a contradiction.

Thanks to the statement we have just proved, we can find a large number $r_0 > 0$ such that the submanifold Σ_0 meets all the spheres $\mathbb{S}^{n-1}(r)$ transversely for $r \geq r_0$. Therefore, when $r = r_0$ such intersection consists of finitely many, say $N_0 = N_0(r_0)$ smooth submanifolds Ξ_1, \dots, Ξ_{N_0} of dimension $n-2$. Furthermore, arguing as we did above for the $n=3$ case the distance squared function $r^2 : \Sigma_0 \cap B_{r_0}^c \rightarrow \mathbb{R}$ cannot have interior critical points for $r > r_0$ and therefore, by basic results in Morse theory $\Sigma_0 \setminus B_{r_0}$ consists of N_0 ends of annular type namely $\Sigma_0 \setminus B_{r_0} \simeq \bigsqcup \Xi_i \times (r_0, +\infty)$. Without loss of generality, let us assume from now onwards that Σ_0 is renamed to be one of those annular ends.

Step 2: expansion at infinity.

Because of the polynomial volume growth assumption, the hypersurface Σ_0 does admit a cone at infinity in the following sense. For any sequence $\lambda_m \searrow 0$ there exists a subsequence (which we do not rename) such that $\lambda_m \Sigma_0 \rightarrow \Gamma$ for some stable minimal hypercone $\Gamma \subset \mathbb{R}^n$. The convergence happens in the sense of integral varifolds (see Chapter 4 of [Sim83]), which is to say (in our special setting) that for any fixed continuous function f with compact support in $\mathbb{R}^n \setminus \overline{B}_{r_0}$ one has $\int_{\{\lambda_m^{-1}x: x \in \Sigma_0\}} f d\mathcal{H}^{n-1} \rightarrow \int_{\Gamma} f d\mathcal{H}^{n-1}$. Due to the fact that the ambient dimension is less than eight we then know that the cone Γ has to be regular, since it is in fact an hyperplane Π . Moreover, by virtue of the previous step, we know that the ends of Σ_0 are annular and can be separated, so that by working one annular end at a time we can certainly assume that Γ has multiplicity one. At this stage, we are then in position to apply Theorem 5.7 in [Sim85] (see also the discussion given at pp. 269-270) which implies that Σ_0 is an outer-graph: there exists a function $u \in \mathcal{C}^2(\Pi \setminus B_{r_0}; \mathbb{R})$ whose graph coincides with Σ_0 and moreover

$$|x|^{-1}|u(x)| + |\nabla_{\Pi} u(x)| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

At that stage, we make use of the information below (together with the fact that $\int_{\Sigma} |A|^2 d\mathcal{H}^{n-1} < \infty$, which is forced by the strong stability assumption) to get for u an expansion of the form

$$u(x') = b + c|x'|^{3-n} + O(|x'|^{2-n})$$

so that Σ_0 is regular at infinity.

Step 3: infinitesimal rigidity.

To prove infinitesimal rigidity in dimension k we will make use of the Positive Mass Theorem in dimension $k-1$. It is first convenient to state and prove the following simple lemma.

Lemma 3.3. *Let (M, g) and Σ be as above. Suppose that there exists a function $\psi \in \mathcal{V}_{\alpha}(\Sigma)$ such that*

$$\int_{\Sigma} \left(|\nabla_{\Sigma} \psi|^2 + \frac{n-3}{4(n-2)} R_{\Sigma} \psi^2 \right) d\mathcal{H}^{n-1} = 0$$

Then $\psi = \alpha$ for \mathcal{L}^{n-1} a.e. $x \in \Sigma$.

Proof. If some function ψ satisfies $\int_{\Sigma} \left(|\nabla_{\Sigma} \psi|^2 + \frac{n-3}{4(n-2)} R_{\Sigma} \psi^2 \right) d\mathcal{H}^{n-1} = 0$, then obviously the integral $\int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1} \leq 0$ and therefore, because of the strong stability assumption, the fact that $c(k) = (k-2)/4(k-1)$ for any $k \geq 3$ and that the term $Q = \frac{1}{2}(R + |A|^2)$ is

non-negative, we know that the following chain of inequalities holds:

$$\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 d\mathcal{H}^{n-1} \geq -\frac{1}{2} \int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1} \geq -c(n-1) \int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1}.$$

As a result, necessarily $\int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1} = 0$ and thus also $\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 d\mathcal{H}^{n-1} = 0$. This forces ψ to be constant \mathcal{H}^{n-1} a.e. on Σ and due to its behaviour at infinity (recall that we assumed $\psi \in \mathcal{V}_{\alpha}$) the conclusion follows. \square

We now derive infinitesimal rigidity for Σ . In view of Step 2 we know that Σ itself can be regarded as an asymptotically flat manifold with an induced Riemannian metric given by

$$\bar{g}_{ij} = g \left(\frac{\partial U}{\partial x^i}, \frac{\partial U}{\partial x^j} \right) = h^{\frac{4}{n-2}} \left(\delta_{ij} + \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \right) + O(|x|^{-(n-1)}), \text{ as } |x| \rightarrow \infty.$$

This is true at each of its ends (u and h are, respectively, the defining function and the conformal factor of the end we are considering). In order to deform \bar{g} to a scalar flat metric on Σ we need to prove that the conformal Laplacian $P_{\Sigma} : \mathcal{W}_{\frac{3-n}{2}}^{1,2}(\Sigma) \rightarrow \mathcal{W}_{\frac{-1-n}{2}}^{-1,2}(\Sigma)$ is an isomorphism. First of all, Lemma 3.3 applied for $\alpha = 0$ implies at once that P_{Σ} has to be injective. At that point, a standard application of the Sobolev inequality [Bar86] gives that for any datum $\theta \in \mathcal{W}_{\frac{-1-n}{2}}^{-1,2}(\Sigma)$ the functional \mathcal{F}_{θ} given by

$$\varphi \mapsto \int_{\Sigma} \left(|\nabla_{\Sigma} \varphi|^2 + \frac{n-3}{4(n-2)} \varphi^2 - \theta \varphi \right) d\mathcal{H}^{n-1}$$

is bounded from below and coercive and thus, following the direct method of the Calculus of Variations (where we exploit the Rellich compactness for these weighted spaces, [Bar86]), we conclude that it must have a critical point $\varphi_0 \in \mathcal{V}_0(\Sigma) = \mathcal{W}_{\frac{3-n}{2}}^{1,2}(\Sigma)$. This completes the proof of the claim. When $\theta = -\frac{n-3}{4(n-2)} R_{\Sigma}$ we thus obtain a function χ with the property that $P_{\Sigma}(1 + \chi) = 0$. If we let $\psi = 1 + \chi$ strong stability and integration by parts give

$$\begin{aligned} -c(n-1) \int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1} &\leq 2c(n-1) \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 d\mathcal{H}^{n-1} \leq \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 d\mathcal{H}^{n-1} \\ &= -c(n-1) \int_{\Sigma} R_{\Sigma} \psi^2 d\mathcal{H}^{n-1} + \lim_{\sigma \rightarrow \infty} \int_{\partial D_{\Sigma}} \psi \nabla_{\eta} \psi d\mathcal{H}^{n-2}. \end{aligned}$$

Since linear theory (see e. g. Meyers [Mey63]) gives, for ψ , an expansion of the form

$$\psi(x') = 1 + \frac{2\mathcal{M}}{n-2} |x'|^{3-n} + O(|x'|^{2-n})$$

we immediately see that previous inequality forces $\mathcal{M} \leq 0$. On the other hand, \mathcal{M} represents the ADM mass of the asymptotically flat manifold $(\Sigma, \psi^{4/(n-3)} \bar{g})$ while the positive mass Theorem 2.6, in dimension $n-1$, gives $\mathcal{M} \geq 0$ and hence in fact $\mathcal{M} = 0$.

At that stage, we can re-consider the previous chain of inequalities and see at once that we are in position to exploit Lemma 3.3 and conclude that $\psi = 1$ identically on Σ . This means that (Σ, \bar{g}) had to be scalar flat.

Step 4: conclusion via Positive Mass Theorem.

By performing computations similar to those we did above in the case $n = 3$ we see that if $\mathcal{M}_j > 0$ then, along that end, $Ric(\nu, \nu) \geq C|x|^n$ while $|R| \leq C|x|^{n+1}$ so that the Gauss

equation gives $|R_\Sigma| \leq -C|x|^{-n}$ as $x \rightarrow \infty$ (for $\{x\}$ a set of asymptotically flat coordinates on the end in question, and $C > 0$). This contradicts the conclusion of Step 3, unless (at least) one end of (M, g) has mass zero, and the conclusion follows from Theorem 2.6. \square

APPENDIX A. IMPROVING DECAY FROM TANGENT CONE UNIQUENESS

We give here the details of an argument we mentioned both in Step 3 of the proof of Theorem 1 and in Step 2 of the proof of Theorem 2. In both cases, we could describe the hypersurface Σ (or one end thereof) as the graph of a function $u \in \mathcal{C}^2(\Pi^* \simeq \mathbb{R}^{n-1} \setminus B; \mathbb{R})$ with the properties that

$$\lim_{x \rightarrow \infty} |\nabla u| = 0, \quad \int_{\Pi} |\nabla \nabla u|^2 d\mathcal{L}^{n-1} < \infty.$$

(The function u is only defined in the complement of an open ball in an hyperplane Π). We remark that the second condition is equivalent to the finiteness of the total curvature of Σ , since the gradient is bounded.

A.1. From Hölder decay of the gradient to optimal decay. In this subsection, we assume that $|\nabla u| \leq C|x'|^{-\alpha}$ (in asymptotically flat coordinates $\{x\}$) for some positive α and we indicate how an asymptotic expansion for u can be obtained. First of all, let us observe that by our assumption we also get $|\nabla \nabla u| \leq C|x'|^{-1-\alpha}$ and that the higher derivatives decay correspondingly. Moreover, based on the fact that we are working with asymptotically flat data, all of the previous statements can be phrased in terms of Euclidean derivatives in asymptotically flat coordinates. A simple computation shows that the function u solves a quasi-linear elliptic problem of the form

$$\sum_{i,j=1}^{n-1} \left(\delta_{ij} - \frac{u_i u_j}{1 + |\partial u|^2} \right) u_{,ij} + 2 \left(\frac{n-1}{n-2} \right) \sqrt{1 + |\partial u|^2} \frac{\partial h}{\partial \nu_0} + \mathcal{R}(x') = 0$$

where $\nu_0 = \frac{(-\partial u, 1)}{\sqrt{1 + |\partial u|^2}}$ is, as usual, the Euclidean unit normal and \mathcal{R} is a remainder term such that

$$|\mathcal{R}(x')| \leq C \left[\frac{|u|}{|x'|^{n+1}} (1 + |\partial u| + |\partial u|^2) + \frac{1}{|x'|^n} (1 + |\partial u| + |x'| |\partial^2 u|) \right].$$

Based on the *a-priori* decay of $\partial u, \partial \partial u$ (and on the fact that, by integration along radii trivially $|u(x')| \leq C|x|^{1-\alpha}$) we can rewrite the equation in the much simpler form

$$\Delta u = f$$

where $|f(x')| \leq C|x|^{\max\{-1-3\alpha, -n+1-\alpha\}}$. Standard PDE theory guarantees that we can solve the problem

$$\begin{cases} \Delta v = f & \text{on } \Pi^* = \mathbb{R}^{n-1} \setminus B \\ v = u & \text{on } \partial \Pi^* = \partial B \end{cases}$$

for some $v(x) \leq |x|^{\max\{1-3\alpha, -n+3-\alpha\}}$ and thus $w = u - v$ is an harmonic function defined on the complement of a ball B and vanishing on $\partial B = \partial\Pi^*$. Since $h(x) = o(|x|)$ as $|x| \rightarrow \infty$, if $n = 3$ we conclude that h grows at most logarithmically and has an expansion of the form

$$w(x) = a_{-1} \log |x| + a_0 + \sum_{n \geq 1} (b_n \cos(n\theta) + b'_n \sin(n\theta)) |x|^{-n}.$$

Similarly, if $n \geq 4$ we conclude that $w(x)$ decays (modulo an additive constant) like $w(x') \leq C|x'|^{3-n}$ and has an expansion. As a result, we obtain the bound $|u(x')| \leq C|x'|^{\max\{1-3\alpha, -n+3-\alpha\}}$ and we can exploit this information in the minimal surface equation solved by u . Iterating this argument finitely many times, we obtain that:

- if $n = 3$ grows at most logarithmically and $u(x') = a + b \log |x'| + O(|x'|^{-1})$ as $|x'| \rightarrow \infty$;
- if $n \geq 4$ decays at a rate $|x'|^{3-n}$ and $u(x') = a + b|x'|^{3-n} + O(|x'|^{-(2-n)})$ as $|x'| \rightarrow \infty$.

A.2. Proving Hölder decay of the gradient. We then move to the preliminary part of the argument consisting in getting a pointwise decay estimate for $|\nabla u|$. By the De Giorgi Lemma (see, for instance, Theorem 5.3.1 in [Mor66]) it is enough, to that aim, to prove an integral estimate of the form

$$\int_{\Pi \setminus B_\sigma} |\partial \partial u|^2 d\mathcal{L}^{n-1} \leq C\sigma^{-2\alpha}$$

for some constant $C > 0$ independent of σ . Let us fix an index $1 \leq k \leq n-1$ and differentiate in x^k the equation solved by the function u : in turn $v_k = u_{,k}$ solves $T(v_k) = \mathcal{R}$ where

$$T(v_k) = \partial_{x^i} (a^{ij} \partial_{x^j} v_k)$$

for $a^{ij} = (\delta^{ij} - \nu_{0i} \nu_{0j}) / \sqrt{1 + |\partial u|^2}$ and $|\mathcal{R}(x')| \leq C|x'|^{-n}$ as $|x'| \rightarrow \infty$. From this equation we see that for any constant vector β we have $|\partial u - \beta|^2 = \sum_k (v_k - \beta_k)^2$ satisfies

$$T(|\partial u - \beta|^2) = 2 \sum_{i,j,k} a^{ij} (\partial_{x^i} \partial_{x^k} u) (\partial_{x^j} \partial_{x^k} u) \geq |\partial \partial u|^2 - C|x'|^{-n}$$

at least for $|x|$ large enough. That being said, for any large σ we choose a cutoff function φ which is one outside $B_{2\sigma}$ and zero inside B_σ . We may multiply by φ^2 and integrate by parts using the integrability of $|\partial \partial u|^2$ to justify the result, thus obtaining

$$\int_{B_{2\sigma} \setminus B_\sigma} -a^{ij} (\partial_{x^i} \varphi^2) (\partial_{x^j} |\partial u - \beta|^2) d\mathcal{L}^{n-1} \geq \int_{\Pi \setminus B_\sigma} \varphi^2 |\partial \partial u|^2 d\mathcal{L}^{n-1} - C \int_{\Pi \setminus B_\sigma} |x'|^{-n}.$$

The standard manipulation (based on Young's inequality) and rearrangement give

$$\int_{\Pi \setminus B_\sigma} \varphi^2 |\partial \partial u|^2 d\mathcal{L}^{n-1} \leq C \int_{B_{2\sigma} \setminus B_\sigma} |\partial \varphi|^2 |\partial u - \beta|^2 d\mathcal{L}^{n-1} + C\sigma^{-1}$$

which of course implies

$$\int_{\Pi \setminus B_{2\sigma}} |\partial \partial u|^2 d\mathcal{L}^{n-1} \leq C\sigma^{-2} \int_{B_{2\sigma} \setminus B_\sigma} |\partial u - \beta|^2 d\mathcal{L}^{n-1} + C\sigma^{-1}.$$

Choosing the vector β to be the average of the gradient ∂u over the annulus and applying the Poincaré-Wirtinger inequality we obtain

$$\int_{\Pi \setminus B_{2\sigma}} |\partial \partial u|^2 d\mathcal{L}^{n-1} \leq C \int_{B_{2\sigma} \setminus B_\sigma} |\partial \partial u|^2 d\mathcal{L}^{n-1} + C\sigma^{-1}.$$

If we denote $J(\sigma) = \int_{\Pi \setminus B_\sigma} |\partial \partial u|^2 d\mathcal{L}^{n-1}$, the previous inequality can be written in the form

$$J(2\sigma) \leq C(J(2\sigma) - J(\sigma)) + C\sigma^{-1}$$

or $J(2\sigma) \leq \theta J(\sigma) + \xi\sigma^{-1}$. At that stage, the claim is proved by means of a version *at infinity* of a general iteration lemma, as stated here.

Let $\sigma \in \mathbb{R}$, $\sigma \geq \sigma_0$ and suppose we know that there exist constants $\theta \in (0, 1)$, $\lambda > 1$ and $\xi > 0$ such that

$$J(\lambda\sigma) \leq \theta J(\sigma) + \xi\sigma^{-1}, \quad \forall \sigma \geq \sigma_0.$$

It is convenient to set $\omega = -\log_\lambda \theta$ and let us notice that $\omega > 0$, while we cannot say, at least a priori, whether $\omega \in (0, 1]$ or instead $\omega > 1$. Let us define $G(\tau) = J(\frac{1}{\tau})$ for $\tau \in (0, \tau_0]$, where clearly $\tau_0 = \sigma_0^{-1}$; our assumption turns into the equivalent form

$$G\left(\frac{\tau}{\lambda}\right) \leq \lambda^{-\omega} G(\tau) + \xi\tau, \quad \forall \tau \in (0, \tau_0]$$

which is the starting point for the following elementary iteration result.

Lemma A.1. *Let $G : (0, \tau_0]$ a non-decreasing function such that*

$$(A.1) \quad G\left(\frac{\tau}{\lambda}\right) \leq \lambda^{-\omega} G(\tau) + \xi\tau, \quad \forall \tau \in (0, \tau_0]$$

for some $\omega > 0$ and $\xi > 0$. Then, there exists a real constant $C = C(\omega, \xi, \lambda, \tau_0)$ such that

$$G(\tau) \leq C\tau^{\overline{\omega}}, \quad \forall \tau \in (0, \tau_0]$$

where we have set $\overline{\omega} = \min\{1, \omega\}$.

Proof. Given $\tau \leq \tau_0$, let $P \in \mathbb{N}$ be the only nonnegative integer such that

$$\frac{\tau_0}{\lambda^{P+1}} < \tau \leq \frac{\tau_0}{\lambda^P}.$$

Now if $P \geq 1$, by our assumption (A.1), we know that

$$G\left(\frac{\tau_0}{\lambda^P}\right) \leq \lambda^{-\omega} G\left(\frac{\tau_0}{\lambda^{P-1}}\right) + \xi \frac{\tau_0}{\lambda^{P-1}}$$

and then, by iteration, an elementary induction argument gives that

$$G\left(\frac{\tau_0}{\lambda^P}\right) \leq \lambda^{-P\omega} G(\tau_0) + \lambda^\omega \frac{\xi \tau_0}{\lambda^P} \sum_{j=1}^P \lambda^{(1-\omega)j}$$

and therefore, since $G(\cdot)$ is nondecreasing, this implies

$$(A.2) \quad G(\tau) \leq \lambda^{-P\omega} G(\tau_0) + \lambda^\omega \frac{\xi \tau_0}{\lambda^P} \sum_{j=1}^P \lambda^{(1-\omega)j}.$$

To proceed further, it is convenient to consider the two cases when $\omega > 1$ or $\omega \in (0, 1]$ separately. In the former, we immediately get from (A.2), by simply replacing the partial sum by the whole series (which is obviously *summable*)

$$G(\tau) \leq \lambda^\omega \left(\frac{\tau}{\tau_0} \right)^\omega G(\tau_0) + C(\omega, \xi, \lambda) \tau$$

and hence

$$G(\tau) \leq C(\omega, \xi, \lambda, \tau_0) \tau, \quad \forall \tau \in (0, \tau_0].$$

In the latter case, it is enough to get an upper bound on such partial sum:

$$\sum_{j=1}^P \lambda^{(1-\omega)j} \leq \frac{\lambda^{1-\omega}}{\lambda^{1-\omega} - 1} \left(\frac{\tau_0}{\tau} \right)^{1-\omega}$$

and hence, from (A.2) we immediately get

$$G(\tau) \leq C(\omega, \xi, \lambda, \tau_0) \tau^\omega, \quad \forall \tau \in (0, \tau_0]$$

which completes the proof. \square

REFERENCES

- [Alm66] F. J. ALMGREN JR., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. **84** (1966), no. 2, 277-292.
- [ADM59] R. ARNOWITT, S. DESER, C. W. MISNER, *Dynamical structure and definition of energy in general relativity*, Phys. Rev. **2** (1959), no. 116, 1322-1330.
- [Bar86] R. BARTNIK, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661-693.
- [Ber17] S. N. BERNSTEIN, *Sur une théorème de géométrie et ses applications aux équations dérivées partielles du type elliptique*, Comm. Soc. Math. Kharkov **15** (1915-1917), 38-45 German translation in Bernstein, S. (1927), "Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus", Mathematische Zeitschrift (Springer) **26**, 551-558.
- [BdGG69] E. BOMBIERI, E. DE GIORGI, E. GIUSTI, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243-268.
- [Car14] A. CARLOTTO, *Rigidity of stable marginally outer-trapped surfaces in initial data sets*, preprint (arXiv: 1404.0358)
- [CS14] A. CARLOTTO, R. SCHOEN, *Localizing solutions of the Einstein constraint equations*, in preparation.
- [CM05] T. COLDING, W. MINICOZZI II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus*, preprint (arXiv: 0509647).
- [CM11] T. COLDING, W. MINICOZZI II, *A Course in Minimal Surfaces*, AMS Graduate studies in Mathematics, 2011.
- [dG65] E. DE GIORGI, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa **19** 1965, no. 3, 79-85.
- [dCP79] M. DO CARMO, C. K. PENG, *Stable complete minimal surfaces in R^3 are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 903-906.
- [EHLS11] M. EICHMAIR, L. H. HUANG, D. A. LEE, R. SCHOEN, *The spacetime positive mass theorem in dimensions less than eight*, preprint (arXiv: 1110.2087).
- [EM12] M. EICHMAIR, J. METZGER, *On large volume preserving stable CMC surfaces in initial data sets*, J. Differential Geom. **91** (2012), no. 1, 81-102.
- [EM13] M. EICHMAIR, J. METZGER, *Large isoperimetric surfaces in initial data sets*, J. Differential Geom. **94** (2013), no. 1, 159-186.
- [FS80] D. FISCHER-COLBRIE, R. SCHOEN, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199-211.

- [Fle62] W. H. FLEMING, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo **11** (1962), no. 2, 69-90.
- [GL86] R. GULLIVER, H. BLAINE LAWSON JR., *The structure of stable minimal hypersurfaces near a singularity*, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Proc. Sympos. Pure Math., 44, Amer. Math. Soc., Providence, RI, 1986, 213-237.
- [MPR13] W. H. MEEKS III, J. PEREZ, A. ROS, *Local removable singularity theorems for minimal laminations*, preprint (arXiv: 1308.6439).
- [Mey63] N. MEYERS, *An expansion about infinity for solutions of linear elliptic equations*, J. Math. Mech. **12** (1963) 247-264.
- [Mor66] C. B. MORREY, *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130 Springer-Verlag New York, Inc., New York 1966 ix+506 pp.
- [Oss69] R. OSSERMAN, *A survey of minimal Surfaces*, Math. Studies 25, Van Nostrand, New York, 1969.
- [Sch83] R. SCHOEN, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791-809
- [Sch83b] R. SCHOEN, *Estimates for stable minimal surfaces in Three Dimensional Manifolds in Seminar on Minimal Submanifolds*, Annals of Mathematiccs Studies , Princeton Uni. Press, vol. 103, (1983).
- [Sch84] R. SCHOEN, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479-495.
- [SS81] R. SCHOEN, L. SIMON, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), no. 6, 741-797.
- [SY79] R. SCHOEN, S. T. YAU, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45-76.
- [Shi85] K. SHIOHAMA, *Total curvatures and minimal areas of complete surfaces*, Proc. Amer. Math. Soc. **94** (1985), no. 2, 310-316.
- [Sim83] L. SIMON, *Lectures on Geometric Measure Theory*, Centre for Mathematical Analysis (Australian National University), 1983.
- [Sim83b] L. SIMON, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525-571.
- [Sim85] L. SIMON, *Isolated singularities of extrema of geometric variational problems. Harmonic mappings and minimal immersions*, Lecture Notes in Math. **1161** pp. 206-277, Springer, Berlin, 1985.
- [Sim68] J. SIMONS, *Minimal varieties in riemannian manifolds*, Ann. of Math. **88** (1968), no. 2, 62-105.
- [Whi87] B. WHITE, *Complete surfaces of finite total curvature*, J. Differential Geom. **26** (1987), no. 2, 315-326.
- [Whi87b] B. WHITE, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), no. 2, 243-256.
- [Wit81] E. WITTEN, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381-402.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305
 E-mail address: carlotto@math.stanford.edu